Quantum Channel Capacity - Handout SEPIA Meeting Adaptive Systems Research Group University of Hertfordshire

Marco Möller

14.11.2007

Dirac Notation in QM (Quantum Mechanic) you normally using the *Dirac Notation*. Vectors are normally column vectors ("*ket*")

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix} = \vec{v} = |v\rangle$$
$$\langle v| = (v_1^*, v_2^*, \ldots) = \vec{v}^{\dagger} = (\vec{v}^*)^T$$
$$\langle v|v\rangle = \langle v| |v\rangle = |\vec{v}|^2$$

- density operator Suppose a quantum system is in one of a number of states $|\psi_i\rangle$, where *i* is an index, with respective probability's p_i .
 - The $|\psi_i\rangle$ are called the pure states.
 - define the density operator

$$\rho \equiv \sum_{i} p_{i} \left| \psi_{i} \right\rangle \left\langle \psi_{i} \right|$$

quantum operation is described by an set of Krausoperator's $\{K_i\}$ with $\sum_i K_i K_i^{\dagger} = I$

$$\rho_{S}^{\prime} = \varepsilon \left(\rho_{S}\right) = \operatorname{tr}_{E} \left(U \left(\rho \otimes \left|a\right\rangle_{E} \left\langle a\right|_{E}\right) U^{\dagger} \right) \\ = \sum_{i} K_{i} \rho_{S} K_{i}^{\dagger}$$

POVM measurement set of Measurement Operators $\{E_m\}$ with $\sum_m E_m = I$. Measurement Result:

$$p(m) = \operatorname{tr}\left(E_m\rho\right)$$

Shannon Entropy $H(X) \equiv -\sum_{x} p_x \log p_x$

Von Neumann Entropy $S(X) \equiv -\text{tr}(\rho \log \rho)$

- Shannon Entropy is special case, iff $\rho = \sum_{x} p_x |x\rangle \langle x|$ and $\langle x|y\rangle = \delta_{xy}$ S(X) = H(X)
- Unit is *bit* resp. *qubit*
- Convention: log with basis 2

Conditional Entropy H(X|Y) = H(X,Y) - H(Y)

Mutual Information

$$H(X:Y) = H(X) + H(Y) - H(X,Y) = H(X) - H(X|Y)$$

- analog for S(X)
- Holevo Bound Suppose Alice prepares a state ρ_X where $X = 0, \ldots, n$ with probability's p_0, \ldots, p_n . Bob performs a measurement described by POVM elements $\{E_y\} = \{E_0, \ldots, E_m\}$ on that state, with measurement outcome Y. The Holevo bound states that for any such measurement Bob may do:

$$H(X:Y) \le S(\rho) - \sum_{x} p_{x} S(\rho_{x}),$$

where $\rho = \sum_{x} p_x \rho_x$.

• If all ρ_x are in a pure state but orthogonal to each other H(X:Y) is maximal.

Fidelity is measure for degree of identity form a quantum operation ϵ with Kraus operators E_i

$$F(\rho, \epsilon) = \sum_{i} \left| \operatorname{tr} \left(\rho E_{i} \right) \right|^{2}$$

i.i.d. the X_i are independent, identical distributed

Law of Large Numbers Suppose X_1, X_2, \ldots i.i.d. with finite first and second moment. Then for any $\epsilon > 0$

$$\lim_{n \to \infty} p\left(\left| \frac{1}{n} \sum_{i=1}^{n} X_i - E\left(X\right) \right| \le \epsilon \right) = 1$$

 ϵ -typical Let $\rho = \sum_{x} p(x) |x\rangle \langle x|$ be an orthonormal decomposition. A sequence x_1, \ldots, x_n is called ϵ -typical, iff

$$\left|\frac{1}{n}\log\left(\frac{1}{p(x_{1})p(x_{2})\cdots p(x_{n})}\right) - S(\rho)\right| \leq \epsilon$$

Correspondingly is the according state $|x_1\rangle |x_2\rangle \cdots |x_n\rangle$ called ϵ -typical.

The subspace of all ϵ -typical states is denoted $T(n, \epsilon)$. The according projector on this subspace is

$$P(n,\epsilon) = \sum_{x \text{ ϵ-typical}} |x_1\rangle \langle x_1| \otimes \cdots \otimes |x_n\rangle \langle x_n|$$

Theorem about Typical Subspaces

1. Consider $\epsilon > 0$. For every $\delta > 0$ and sufficient big n:

$$\operatorname{tr}\left(P\left(n,\epsilon\right)\rho^{\otimes n}\right) \geq 1-\delta$$

2. For every $\epsilon > 0$ and $\delta > 0$ and sufficient big *n* the dimension of $T(n, \epsilon)$ fulfill $|T(n, \epsilon)| = \operatorname{tr} (P(n, \epsilon))$

$$(1-\delta) 2^{n(S(\rho)-\epsilon)} \le |T(n,\epsilon)| \le 2^{n(S(\rho)+\epsilon)}$$

3. Let S(n) be a projector to an arbitrary subspace of $H^{\otimes n}$ with dimension smaller then 2^{nR} . Consider $R < S(\rho)$. Then for all $\delta > 0$ and sufficient big n

$$\operatorname{tr}\left(S\left(n\right)\rho^{\otimes n}\right) \leq \delta$$

- Schumacher's quantum noiseless channel coding Let $\{H, \rho\}$ be an i.i.d. quantum source. If $R > S(\rho)$ then there exists a reliable compression scheme of rate R for the source. If $R < S(\rho)$ then any compression scheme or rate R is not reliable.
 - Reliable corresponds to $F(\rho, \cdot) \to 1$ with $n \to \infty$
- **Shannon: noisy coding** For a noisy channel \mathcal{N} the capacity is given by

$$\mathcal{C}\left(\mathcal{X}\right) = \max_{p(x)} H\left(X:Y\right)$$

where maximum is to taken over all possible input distributions p(x) for X and Y is the corresponding output random variable at the output of the channel.

- Channel is described by a set of conditional probability's $p(x|y) \ge 0$
- In qm it is much more complicated \Rightarrow Holevo-Schumacher-Westmoreland Theorem