Quantum Channel Capacity - Handout
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Dirac Notation in QM (Quantum Mechanic) you normally using the *Dirac Notation*. Vectors are normally column vectors ("*ket*")

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix} = \vec{v} = |v\rangle$$

$$\langle v| = (v_1^*, v_2^*, \ldots) = \vec{v}^\dagger = (\vec{v}^*)^T$$

$$\langle v|v\rangle = \langle v|\,|v\rangle = |\vec{v}|^2$$

density operator Suppose a quantum system is in one of a number of states $|\psi_i\rangle$, where i is an index, with respective probability's p_i.

- The $|\psi_i\rangle$ are called the pure states.
- define the density operator

$$\rho \equiv \sum_i p_i |\psi_i\rangle \langle\psi_i|$$

quantum operation is described by an set of Kraus-operator' s $\{K_i\}$ with $\sum_i K_i K_i^\dagger = I$

$$\begin{aligned} \rho_S' = \varepsilon(\rho_S) &= \mathrm{tr}_E \left(U \left(\rho \otimes |a\rangle_E \langle a|_E \right) U^\dagger \right) \\ &= \sum_i K_i \rho_S K_i^\dagger \end{aligned}$$

POVM measurement set of Measurement Operators $\{E_m\}$ with $\sum_m E_m = I$. Measurement Result:

$$p(m) = \mathrm{tr}(E_m \rho)$$

Shannon Entropy $H(X) \equiv -\sum_x p_x \log p_x$

Von Neumann Entropy $S(X) \equiv -\mathrm{tr}(\rho \log \rho)$

- Shannon Entropy is special case, iff $\rho = \sum_x p_x |x\rangle\langle x|$ and $\langle x|y\rangle = \delta_{xy}$
  $S(X) = H(X)$
- Unit is *bit* resp. *qubit*
- Convention: log with basis 2

Conditional Entropy $H(X|Y) = H(X,Y) - H(Y)$

Mutual Information

$$\begin{aligned} H(X:Y) &= H(X) + H(Y) - H(X,Y) \\ &= H(X) - H(X|Y) \end{aligned}$$

- analog for $S(X)$

Holevo Bound Suppose Alice prepares a state ρ_X where $X = 0, \ldots, n$ with probability's $p_0, \ldots, p_n$. Bob performs a measurement described by POVM elements $\{E_y\} = \{E_0, \ldots, E_m\}$ on that state, with measurement outcome Y. The Holevo bound states that for any such measurement Bob may do:

$$H(X:Y) \le S(\rho) - \sum_x p_x S(\rho_x),$$

where $\rho = \sum_x p_x \rho_x$.

- If all ρ_x are in a pure state but orthogonal to each other $H(X:Y)$ is maximal.

Fidelity is measure for degree of identity form a quantum operation ϵ with Kraus operators E_i

$$F(\rho, \epsilon) = \sum_i |\mathrm{tr}(\rho E_i)|^2$$

i.i.d. the X_i are **i**ndependent, **i**dentical **d**istributed

Law of Large Numbers Suppose $X_1, X_2, \ldots$ i.i.d. with finite first and second moment. Then for any $\epsilon > 0$

$$\lim_{n\to\infty} p\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - E(X)\right| \leq \epsilon\right) = 1$$

ϵ-typical Let $\rho = \sum_x p(x)|x\rangle\langle x|$ be an orthonormal decomposition. A sequence $x_1, \ldots, x_n$ is called ϵ-typical, iff

$$\left|\frac{1}{n}\log\left(\frac{1}{p(x_1)\,p(x_2)\cdots p(x_n)}\right) - S(\rho)\right| \leq \epsilon$$

Correspondingly is the according state $|x_1\rangle|x_2\rangle\cdots|x_n\rangle$ called ϵ-typical.
The subspace of all ϵ-typical states is denoted $T(n,\epsilon)$. The according projector on this subspace is

$$P(n,\epsilon) = \sum_{x\ \epsilon\text{-typical}} |x_1\rangle\langle x_1| \otimes \cdots \otimes |x_n\rangle\langle x_n|$$

Theorem about Typical Subspaces

1. Consider $\epsilon > 0$. For every$\delta > 0$ and sufficient big n:
$$\mathrm{tr}\left(P(n,\epsilon)\,\rho^{\otimes n}\right) \geq 1 - \delta$$
2. For every $\epsilon > 0$ and $\delta > 0$ and sufficient big n the dimension of $T(n,\epsilon)$ fulfill $|T(n,\epsilon)| = \mathrm{tr}(P(n,\epsilon))$
$$(1-\delta)\,2^{n(S(\rho)-\epsilon)} \leq |T(n,\epsilon)| \leq 2^{n(S(\rho)+\epsilon)}$$
3. Let $S(n)$ be a projector to an arbitrary subspace of $H^{\otimes n}$ with dimension smaller then 2^{nR}. Consider $R < S(\rho)$. Then for all $\delta > 0$ and sufficient big n
$$\mathrm{tr}\left(S(n)\,\rho^{\otimes n}\right) \leq \delta$$

Schumacher's quantum noiseless channel coding Let $\{H, \rho\}$ be an i.i.d. quantum source. If $R > S(\rho)$ then there exists a reliable compression scheme of rate R for the source. If $R < S(\rho)$ then any compression scheme or rate R is not reliable.

- Reliable corresponds to $F(\rho, \cdot) \to 1$ with $n \to \infty$

Shannon: noisy coding For a noisy channel $\mathcal{N}$ the capacity is given by

$$\mathcal{C}(\mathcal{X}) = \max_{p(x)} H(X:Y)$$

where maximum is to taken over all possible input distributions $p(x)$ for X and Y is the corresponding output random variable at the output of the channel.

- Channel is described by a set of conditional probability's $p(x|y) \geq 0$
- In qm it is much more complicated $\Rightarrow$ Holevo-Schumacher-Westmoreland Theorem